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AVERAGING IN PROBLEMS OF THE BENDING AND OSCILLATION OF STRESSED INHOMOGENEOUS PLATES*

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A method for describing on the average the bending and oscillation of strongly inhomogeneous plates, stressed in their plane, is proposed. A problem that arises in various fields of engineering differs from those considered in /2-4/ in that the operators are not known a priori to be of fixed sign.

1. The bending of an inhomogeneous stressed plate. We consider a plate with irregular thickness of irregular elastic constants (ribbed or composition). Let forces be applied to the plate that produce in its plane a stressed state $\sigma_{ij}^\varepsilon(\mathbf{x})$ (the parameter ε characterizes the degree of irregularity). In the context of the Kirchhoff-Love hypothesis, the equation of equilibrium may be written as ($w^\varepsilon(\mathbf{x})$ is the normal bending of the plate) /1, 5/

$$\begin{aligned} -L_\varepsilon w^\varepsilon + M_\varepsilon w^\varepsilon \equiv {} & [D^\varepsilon(w^\varepsilon_{,11} + \nu^\varepsilon w^\varepsilon_{,22})]_{,11} + 2[D^\varepsilon(1-\nu^\varepsilon)w^\varepsilon_{,12}]_{,12} + \\ & [D^\varepsilon(w^\varepsilon_{,22} + \nu^\varepsilon w^\varepsilon_{,11})]_{,22} - [\sigma_{11}^\varepsilon w^\varepsilon_{,1}]_{,1} - [\sigma_{12}^\varepsilon w^\varepsilon]_{,2} - [\sigma_{21}^\varepsilon w^\varepsilon_{,2}]_{,1} - \\ & [\sigma_{22}^\varepsilon w^\varepsilon_{,2}]_{,2} = f(\mathbf{x}) \end{aligned} \tag{1.1}$$

The flexural rigidity $D^\varepsilon(\mathbf{x})$ and Poisson's ratio $\nu^\varepsilon(\mathbf{x})$ (we consider locally isotropic plates) depend on the space variable $\mathbf{x} \in Q$; $Q \subset R^2$ is the bounded domain occupied by the plate. As the dependence of D^ε, ν^ε on $\mathbf{x}$, we take /2, 6, 7/ $D^\varepsilon(\mathbf{x}) = D(\mathbf{x}/\varepsilon)$, $\nu^\varepsilon(\mathbf{x}) = \nu(\mathbf{x}/\varepsilon)$, where the functions $D(\mathbf{y})$, $\nu(\mathbf{y})$ have the characteristic size of oscillation equal to unity. The stresses $\sigma_{ij}^\varepsilon(\mathbf{x})$ in the plane of the plate are also functions of $\mathbf{x}$ with the characteristic size of oscillation equal to the characteristic size of the irregularity ε. For $\varepsilon \ll 1$, i.e., in the case of strongly irregular plates, in order to describe the bending and loss of stability we use /2-4, 8/ the asymptotic method of homogenization /6, 7/.

Problem (1.1) will be studied asymptotically as $\varepsilon \to 0$ with the proviso that the plate edges are rigidly clamped (we know /1, 2/ that this is equivalent to considering (1.1) in functional space $H_0^2(Q)$ /9, 10/). We consider the problem in the abstract statement. Given the sequences of linear selfadjoint operators, bounded uniformly as $\varepsilon \to 0$,

$$L_\varepsilon, L: H_0^2(Q) \to H^{-2}(Q); \quad M_\varepsilon, M: H_0^k(Q) \to H^{-k}(Q), \quad 0 \leqslant k < 2 \tag{1.2}$$

(for the definition of a space of type $H^\alpha(Q)$ see e.g., /9, 10/). The operator $-L_\varepsilon$ is the sum of the first three terms of the left-hand side of (1.1), while $-M_\varepsilon$ is the sum of the remaining terms, which describe the influence of the stresses in the plane of the plate on its normal bending. Let the operators L_ε and L be positive definite: there exists $c > 0$, independent of $\varepsilon \to 0$, such that $\langle L_\varepsilon u, u\rangle_2 \geqslant c\|u\|_2^2$ for any $u \in H_0^2(Q)$ ($\langle\cdot,\cdot\rangle_k$, $\|\cdot\|_k$ is the operation of pairing and norming in $H_0^k(Q)$ /9/).

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This condition is satisfied if there exists $c_1 > 0$, independent of $\varepsilon \to 0$, such that $c_1 \leqslant D^\varepsilon(x) \leqslant 1/c_1$, $c_1 \leqslant 1 - \nu^\varepsilon(x)$ (e.g., if the material characteristics and thickness of the plate lie in an interval, independent of $\varepsilon \to 0$). Notice that ε can be understood to be either a continuous nor a discrete parameter (e.g., $\varepsilon = 1/n$, $n \in N$).

Definition 1 /11/. The sequence of operators A_ε: $H_0^m(Q) \to H^{-m}(Q)$ is *G*-convergent to the operator A: $H_0^m(Q) \to H^{-m}(Q)$, if, for any $v^* \in H^{-m}(Q)$ $A_\varepsilon^{-1}v^* \to A^{-1}v^*$ weakly in $H_0^m(Q)$ as $\varepsilon \to 0$.

Definition 2 /12/. The sequence of operators A_ε: $H_0^m(Q) \to H^{-m}(Q)$ is strongly convergent in $H^{-m}(Q)$ to the operator A: $H_0^m(Q) \to H^{-m}(Q)$, if, for any $u \in H_0^m(Q)$ $A_\varepsilon u \to Au$ in $H^{-m}(Q)$ as $\varepsilon \to 0$.

Proposition 1. Let the sequence (1.2) of operators L_ε be *G*-convergent to the operator *L*, and the sequence (1.2) of operators M_ε be strongly convergent in $H^{-k}(Q)$ to the operator *M*. Consider the problem

$$-Lw + Mw = f \tag{1.3}$$

Provided that $\lambda = 1$ is not an eigenvalue of the problem $Lw = \lambda Mw$ (condition A), the sequence of soltuions of problem (1.1) $w^\varepsilon \to w$ weakly in $H_0^2(Q)$ as $\varepsilon \to 0$, where w is a solution of problem (1.3).

For the proof we require some preliminary lemmas.

Lemma 1. If the sequences of operators L_ε, M_ε are convergent to operators L, M in the sense of Proposition 1, then, if condition A holds, there exists a number $c_2 > 0$, independent of $\varepsilon \to 0$, such that

$$\operatorname{dist}(\{1\}, \mathrm{Sp}_\varepsilon) \equiv \inf_{x \in \mathrm{Sp}_\varepsilon} |1 - x| \geqslant c_2 \tag{1.4}$$

where Sp_ε is the spectrum of the problem $L_\varepsilon w = \lambda M_\varepsilon w$.

Proof. 1°. We shall show that the *G*-convergence of the sequence of operators L_ε: $H_0^2(Q) \to H^{-2}(Q)$ to the operator *L* implies that

$$\| L_\varepsilon^{-1} - L^{-1} \|_{-l, l} = o(1), \quad 0 < l < 2 \tag{1.5}$$

(For operators $\| \cdot \|_{\alpha,\beta}$ we put $\| \cdot \|_{H^\alpha \to H^\beta}$.)

The operators $L_\varepsilon^{-1}, L^{-1}$: $H^{-2}(Q) \to H_0^2(Q)$ exists by virtue of the conditions imposed on L_ε, L, and hence they are defined as operators of $H^{-l}(Q) \subset H^{-2}(Q)$ and $H_0^l(Q) \supset H_0^2(Q)$. If relation (1.5) is violated, there will exist $\delta > 0$ and $\{u_\varepsilon^*\} \subset H^{-l}(Q)$ such that $\| u_\varepsilon^* \|_{-l} = 1$ and

$$\| L_\varepsilon^{-1}u_\varepsilon^* - L^{-1}u_\varepsilon^* \|_l \geqslant \delta \tag{1.6}$$

Since $\| u_\varepsilon^* \|_{-2} \leqslant \| u_\varepsilon^* \|_{-l} \leqslant 1$ and the imbedding of $H^{-l}(Q)$ in $H^2(Q)$ is compact /10, p.123/, there is a subsequence $\{u_\eta^*\} \subset \{u_\varepsilon^*\}$, such that $u_\eta^* \to u^*$ in $H^{-2}(Q)$ as $\eta \to 0$. Further, given any $u^* \in H^{-2}(Q)$, the sequence $\| L_\varepsilon^{-1}u^* \|_l \to \| L^{-1}u^* \|_l < \infty$ by Definition 1. Then, by the theorem on uniform boundedness /9, p.269/, the sequence $\| L_\varepsilon^{-1} \|_{-2, l}$ is uniformly bounded as $\varepsilon \to 0$. Then,

$$\| L_\eta^{-1}u_\eta^* - L^{-1}u_\eta^* \|_l \leqslant \| L_\eta^{-1} \|_{-2, l} \| u_\eta^* - u^* \|_{-2} + \| L^{-1} \|_{-2, l} \| u_\eta^* - u^* \|_{-2} + \| L_\eta^{-1}u^* - L^{-1}u^* \|_l \tag{1.7}$$

The expression on the right-hand side of (1.7) tends to zero as $\eta \to 0$ (we use Definition 1), which is a contradiction with (1.6), so that (1.5) is proved.

2°. We shall show that the strong convergence of the sequence of operators M_ε to M in $H^{-k}(Q)$ implies that, with $l > k$,

$$\| M_\varepsilon - M \|_{l, -l} = o(1) \tag{1.8}$$

The proof can be by similar methods to those used in 1°. In fact, if (1.8) is violated, there will be a number $\delta > 0$ and a sequence $\{u_\varepsilon\}, \{v_\varepsilon\} \subset H_0^l(Q)$: $\| u_\varepsilon \|_l, \| v_\varepsilon \|_l = 1$, such that $| \langle M_\varepsilon u_\varepsilon - Mu_\varepsilon, v_\varepsilon \rangle_l | \geqslant \delta$. Noting that the imbedding of space $H^\alpha(Q)$ into $H^\beta(Q)$ is compact for $\infty > \alpha > \beta > -\infty$ /10/ and that the $\| M_\varepsilon \|_{k, -l}$ are uniformly bounded as $\varepsilon \to 0$ /9/ (see Sect.1°), we can extract subsequencies $\{u_\eta\}, \{v_\eta\}$ such that $| \langle M_\eta u_\eta - Mu_\eta, v_\eta \rangle_l | \to 0$ as $\eta \to 0$. This is a contradiction, so that (1.8) is proved.

3°. We rewrite the problem $L_\varepsilon w = \lambda M_\varepsilon w$ in the equivalent form

$$w = \lambda L_\varepsilon^{-1} M_\varepsilon w \tag{1.9}$$

Notice that the operator $L_\varepsilon^{-1}M_\varepsilon$: $H_0^l(Q) \to H_0^l(Q)$, $2 > l > k$ is selfadjoint, and since the operator L_ε^{-1}: $H^{-2}(Q) \to H_0^2(Q)$ is bounded, is compact /9/. By (1.5) and (1.8), and the uniform boundedness as $\varepsilon \to 0$ of the quantities $\| L_\varepsilon^{-1} \|_{-l, l}, \| M_\varepsilon \|_{l, -l}$, we find that

$$\| L_\varepsilon^{-1}M_\varepsilon - L^{-1}M \|_{l, l} = o(1) \tag{1.10}$$

From (1.10), using /12, p.365/, we obtain

$$\sup_{\lambda \in \mathrm{Sp}} \operatorname{dist}(\lambda, \mathrm{Sp}_\varepsilon) = o(1), \quad \sup_{\lambda_\varepsilon \in \mathrm{Sp}_\varepsilon} \operatorname{dist}(\lambda_\varepsilon, \mathrm{Sp}) = o(1) \tag{1.11}$$

4°. Let (1.4) be violated. Then, there is a subsequence $\{\lambda_\eta: \lambda_\eta \in \mathrm{Sp}_\eta\}$ such that $\lambda_\eta \to 1$ as $\eta \to 0$. By (1.11), there is a subsequence $\{\lambda^\eta: \lambda^\eta \in \mathrm{Sp}\}$ (Sp_ε, Sp are the spectra of problems

$L_\varepsilon w = \lambda M_\varepsilon w$ and $Lw = \lambda Mw$ respectively), such that $|\lambda_\eta - \lambda^\eta| \to 0$, $\eta \to 0$. Thus, $\lambda^\eta \to 1$ as $\eta \to 0$, $\{\lambda^\eta\} \subset \mathrm{Sp}$. But then, $\lambda = 1$ is not a limit point of the spectral set Sp of the compact operator $L^{-1}M$ /9/. Then, $\lambda^\eta \to 1$ as $\eta \to 0$ can occur only if $\lambda^\eta = 1$, starting from some number. But $\lambda = 1$ does not belong to the spectrum Sp of operator $L^{-1}M$, by virtue of condition A. This contradiction proves (1.4).

Lemma 2. Under the conditions of Proposition 1, the sequence of solutions $\{w^\varepsilon\}$ of problem (1.1) is bounded in $H_0^2(Q)$ uniformly as $\varepsilon \to 0$.

Proof. We rewrite (1.1) in the equivalent form

$$w^\varepsilon = L_\varepsilon^{-1} M_\varepsilon w^\varepsilon - L_\varepsilon^{-1} f \tag{1.12}$$

and consider (1.12) in the space $H_0^l(Q)$, $2 > l > k$. Obviously, the solution of (1.12) belonging to space $H_0^l(Q)$, also belongs to $H_0^2(Q)$ and is a solution of problem (1.1). In view of (1.4), for the operator $L_\varepsilon^{-1}M_\varepsilon$ we have the obvious estimate (see e.g., /12/)

$$\|(E - L_\varepsilon^{-1}M_\varepsilon)^{-1}\|_{l,l} \leqslant 1/c_2$$

in view of which $\|w^\varepsilon\|_l \leqslant \|L_\varepsilon^{-1}\|_{-l,l} \|f\|_{-l}/c_2 \leqslant c_6$, where $c_6 < \infty$ is independent of $\varepsilon \to 0$. Since the operators L_ε are uniformly positive definite as $\varepsilon \to 0$, the operators L_ε^{-1}: $H^{-2}(Q) \to H_0^2(Q)$ are uniformly bounded as $\varepsilon \to 0$. The operators M_ε: $H_0^k(Q) \to H^{-k}(Q)$ are uniformly bounded as $\varepsilon \to 0$ by hypothesis. Hence we find that $\|w^\varepsilon\|_2 \leqslant c_7(\|w^\varepsilon\|_l + 1) \leqslant c_8$, where $c_8 < \infty$ is independent of $\varepsilon \to 0$.

Proof of Proposition 1. By lemma 2, the sequence $\{w^\varepsilon\}$ is weakly compact in $H_0^2(Q)$ /9/. Then, there is a subsequence $\{w^\eta\} \in \{w^\varepsilon\}$ such that $w^\eta \to w$ weakly in $H_0^2(Q)$, and strongly in $H_0^l(Q), l < 2$ as $\eta \to 0$. By (1.10), $L_\eta^{-1}M_\eta w^\eta \to L^{-1}Mw$ in $H_0^l(Q)$ as $\eta \to 0$. In view of this and the fact that $L_\eta^{-1}f \to L^{-1}f$ in $H_0^l(Q), l < 2$, as $\eta \to 0$ (by virtue of the G-convergence of the sequence of operators L_ε to the operator L), we see that $w \in H_0^2(Q)$ and is a solution of problem (1.3). This solution is unique, by condition A. Then, $w^\varepsilon \to w$ weakly in $H_0^2(Q)$ as $\varepsilon \to 0$, where w is the solution of problem (1.3).

Corollary. Under the conditions of Proposition 1, $w^\varepsilon \to w$ in $C(\bar{Q})$ as $\varepsilon \to 0$.

For the proof, it suffices to use Proposition 1 and the appropriate imbedding theorem (see e.g., /9/).

The G-limiting operator L, which is again an operator of plate bending (but now homogeneous and in general orthotropic) can be evaluated by the methods of /2, 6, 7/. The operator M, which describes the stressed state in the plane of the averaged plate, can be found as follows. Let the stresses $\sigma_{ij}^\varepsilon(\mathbf{x})$ be found by solving the problem of theory of elasticity of an inhomogeneous plane body /13/, which as the elastic constants $E^\varepsilon(\mathbf{x})$, $\nu^\varepsilon(\mathbf{x})$, satisfying the conditions: $c_9 \leqslant E^\varepsilon(\mathbf{x}) \leqslant 1/c_9$, $-1/2 + c_9 \leqslant \nu^\varepsilon(\mathbf{x}) \leqslant 1 - c_9$, where $c_9 > 0$ is independent of $\varepsilon \to 0$. Then, see /11, 14/, in the case of the first boundary value problem $\sigma_{ij}^\varepsilon(\mathbf{x}) \to \sigma_{ij}(\mathbf{x})$ weakly in $L_2(Q)$ as $\varepsilon \to 0$, where $\sigma_{ij}(\mathbf{x})$ are the stresses, which are found by solving the G-limiting problem of the plane theory of elasticity. The method of obtaining this G-limiting problem is described in /6, 7, 14, 15/.

Proposition 2. If the stresses $\sigma_{ij}^\varepsilon(\mathbf{x})$ in the plane of the plate are found by solving the plane problem of the theory of elasticity of an inhomogeneous body, then the conditions stated in (1.2) and in Proposition 1 on the sequence of operators M_ε are satisfied, if we put $2 > k \geqslant 1, 5$. Here,

$$M = \frac{\partial}{\partial x_j} \sigma_{ij}(\mathbf{x}) \frac{\partial}{\partial x_j}$$

The proof, which is based on the weak convergence of $\sigma_{ij}^\varepsilon(\mathbf{x})$ to $\sigma_{ij}(\mathbf{x})$ in $L_2(Q)$ as $\varepsilon \to 0$ is given in /3/.

Note 1. Problem (1.1) has been considered above in the space $H_0^2(Q)$, i.e., under the assumption that the plate edge is rigidly clamped. The results remain true if the edge is hinged or freely supported, or a combination of these cases /3/. The same applies to the problem of the theory of elasticity finding the stress $\sigma_{ij}^\varepsilon(\mathbf{x})$ in the plane of the plate (/3, Proposition 5/).

2. Natural oscillations of an inhomogeneous stressed plate. The problem of finding the natural frequencies ω_k^ε and natural shapes $w_k^\varepsilon(\mathbf{x})$ of the plate oscillations has the form /1, 5, 13/

$$-L_\varepsilon w_k^\varepsilon + M_\varepsilon w_k^\varepsilon = \omega_k^\varepsilon \rho^\varepsilon(\mathbf{x}) w_k^\varepsilon \tag{2.1}$$

where $0 < c_{10} \leqslant \rho^\varepsilon(\mathbf{x}) \leqslant 1/c_{10}$ (c_{10} is independent of $\varepsilon \to 0$) is the specific mass of the plate. The operators L_ε and M_ε are defined above. We consider the problem in the function space $H_0^2(Q)$ (Note 1 remains true). We define the operator

$$N_\varepsilon : u \in H_0^l(Q) \to \rho^\varepsilon(\mathbf{x}) u \in L_2(Q) \subset H^{-l}(Q)$$

The operator N_ε is uniformly bounded with respect to $\varepsilon \to 0$, as an operator from $H_0^l(Q)$,

$l \geqslant 0$ in $L_2(Q)$. We write problem (2.1) as

$$w_k^\varepsilon = \omega_k^\varepsilon (-L_\varepsilon + M_\varepsilon)^{-1} N_\varepsilon w_k^\varepsilon \tag{2.2}$$

Lemma 3. Under condition A, the sequence of operators $-L_\varepsilon + M_\varepsilon : H_0^2(Q) \to H^{-2}(Q)$ is G-convergent as $\varepsilon \to 0$ to the operator $-L + M : H_0^2(Q) \to H^{-2}(Q)$.

The proof follows from Proposition 1 and Definition 1.

Let the sequence of functions $\rho^\varepsilon(x) \in L_\infty(Q)$ have an average in the sense that there is a function $\langle\rho\rangle(x) \in L_\infty(Q)$ such that $\int_Q (\rho^\varepsilon(x) - \langle\rho\rangle(x)) u(x)\,dx \to 0$, $\varepsilon \to 0$ for any $u \in L_2(Q)$. The operator N is defined in the same way as N_ε, with $\rho^\varepsilon(x)$ replaced by $\langle\rho\rangle(x)$.

Lemma 4. For $l > 0$,

$$\| N_\varepsilon - N \|_{l,-l} = o(1). \tag{2.3}$$

Proof. Note that, by the weak convergence of the sequence of functions $\rho^\varepsilon(x)$ to $\langle\rho\rangle(x)$ in $L_2(Q)$, the sequence of operators N_ε: $H_0^l(Q) \to H^{-l}(Q)$ is convergent to the operator N strongly in $H^{-l}(Q)$ for any $l > 0$ /3/. The lemma then follows from Para.2 of Lemma 1.

We consider problem (2.2) with $2 > l > k \geqslant 1.5$, when Propositions 1, 2 hold.

Lemma 5. For $2 > l > k \geqslant 1.5$, under condition A, we have

$$\| (-L_\varepsilon + M_\varepsilon)^{-1} N_\varepsilon - (-L + M)^{-1} N \|_{l,l} = o(1) \tag{2.4}$$

The proof uses the same methods as in Lemma 1. We make use of Lemma 4.

Proposition 3. Under condition A, we have the following asymptotic behaviour of the natural frequencies $\{\omega_k^\varepsilon\} = \Omega_\varepsilon$ of the plate oscillations:

$$\sup_{\omega^\varepsilon \in \Omega_\varepsilon} \operatorname{dist}(\omega^\varepsilon, \Omega) = o(1), \quad \sup_{\omega \in \Omega} \operatorname{dist}(\omega, \Omega_\varepsilon) = o(1)$$

where $\{\omega_k\} = \Omega$ are the natural frequencies found by solving the problem (which describes the natural oscillations of a homogeneous plate)

$$-Lw_k + Mw_k = \omega_k \langle\rho\rangle(x) w_k \tag{2.5}$$

In view of (2.4), the proof follows at once from /12, p.365/. It has to be borne in mind that the operators $(-L_\varepsilon + M_\varepsilon)^{-1} N_\varepsilon$ and $(-L + M)^{-1} N$ are compact and selfadjoint.

Note 2. All the above results extend to the case of the bending and oscillation of an inhomogeneous beam $(Q = [a, b])$.

Example. To illustrate the application of our results, consider the example of the oscillations of an inhomogeneous beam (see Note 2). In this case

$$L_\varepsilon = -\frac{d^2}{dx^2}\left(D^\varepsilon(x) \frac{d^2}{dx^2}\right), \quad M_\varepsilon = \frac{d}{dx}\left(\sigma \frac{d}{dx}\right) \tag{2.6}$$

The force σ along the beam axis is $(A - B)/[(a - b)\langle 1/E^\varepsilon \rangle]$, where A and B are the displacements of the beam ends with coordinates a, b.

Note 3. In the present case the G-limit of the sequence of operators L_ε exists, if the function $1/D^\varepsilon(x)$ has a weak limit in $L_2([a, b])$ as $\varepsilon \to 0$ (possibly with probability unity) /16/.

Put $D = 1/\langle 1/D^\varepsilon \rangle$, where $\langle\cdot\rangle$ is the operation of taking the weak limit in $L_2([a, b])$. We consider for simplicity the case when D is independent of x (which is the case when $D^\varepsilon(x)$ is a periodic function), Then /16/, $L = -D d^4/dx^4$. Problem (2.5) here takes the form

$$D \frac{d^4 w_k}{dx^4} - \sigma \frac{d^2 w_k}{dx^2} = \omega_k \langle\rho\rangle w_k$$

Assume that the beam, while made of homogeneous material, has a variable cross-sectional area (is ribbed or has cuts /1/). In the case of a ribbed beam $D = E_0 \langle h_\varepsilon^{-3} \rangle^{-1} / (12(1 - \nu^2))$, $\langle 1/E^\varepsilon \rangle = (1/E_0)\langle h_\varepsilon^{-1} \rangle$, where $h_\varepsilon(x)$ is the beam thickness, E_0 is Young's modulus of the material. If the support is hinged, the natural frequencies of oscillation of the averaged beam can be shown to be

$$\left\{\omega_k = \frac{\pi^2 k^2 E_0 (\varkappa k^2 - \langle h_\varepsilon^{-1} \rangle^{-1} (B - A)/(b - a))}{(b - a)^4 \langle\rho\rangle}, \quad k \in N\right\} = \Omega \tag{2.7}$$

$$\varkappa = \langle h_\varepsilon^{-3} \rangle^{-1} \pi^2 / [12(1 - \nu^2)]$$

Condition A takes the form

$$\varkappa k^2/(b - a) \neq \langle h_\varepsilon^{-1} \rangle^{-1}, \quad \forall k \in N \tag{2.8}$$

By Proposition 3 and Notes 1, 2, the spectrum of the natural frequencies of oscillation of the initial inhomogeneous beam converges under condition (2.8) to spectrum (2.7) in the sense indicated in Proposition 3.

3. Forced oscillations of an inhomogeneous stressed plate.

We consider the problem of the oscillation of these plates /1, 5, 13/

$$-L_\varepsilon w^\varepsilon + M_\varepsilon w^\varepsilon = \rho^\varepsilon(\mathbf{x})\,\partial^2 w^\varepsilon/\partial t^2 + f(\mathbf{x}) \tag{3.1}$$

$$w^\varepsilon(\mathbf{x}, 0) = w_0(\mathbf{x}), \quad \partial w^\varepsilon/\partial t\,(\mathbf{x}, 0) = w_1(\mathbf{x}),\ w_0, w_1 \in L_2(Q) \tag{3.2}$$

Formalization and solvability of problem (3.1), (3.2) are connected with the positive definiteness (PD) of its stationary part $L_\varepsilon - M_\varepsilon$ /10, 17/. By hypothesis, operator L_ε are positive definite uniformly with respect to $\varepsilon \to 0$. In the case of homogeneous plates, it suffices, for PD of operators $L_\varepsilon - M_\varepsilon$, to limit the discussion to the class of loads which lead to non-positive definiteness of the stress tensor in the plate plane /13/. In an inhomogeneous plate, however, the stresses $\sigma_{ij}^\varepsilon(\mathbf{x})$ in its plane, found by solving the problem of the theory of the elasticity of an inhomogeneous plane body, are stresses of general type and our method cannot be used. Let us show that, for operators $L_\varepsilon - M_\varepsilon$ to be PD uniformly, starting with some $\varepsilon' > 0$, it suffices to refer the condition for non-positive definiteness to the averaged stress tensor $\sigma_{ij}(\mathbf{x})$ (which defines the operator M, see Proposition 2).

We introduce the sphere $S(1) = \{u \in H_0^2(Q) : \|u\|_2 \leqslant 1\}$.

Lemma 6. Given any $\delta > 0$, there exists $\varepsilon_0(\delta) > 0$ such that, for all $u \in S(1)$ and $\varepsilon \leqslant \varepsilon_0(\delta)$, we have

$$\langle L_\varepsilon u, u\rangle_2 - \langle M_\varepsilon u, u\rangle_2 \geqslant c - \langle Mu, u\rangle_2 - \delta \tag{3.3}$$

Proof. In view of the conditions imposed on L_ε: $\langle L_\varepsilon u, u\rangle_2 \geqslant c,\ \forall u \in S(1)$. Since the imbedding of $H_0^2(Q)$ into $H_0^k(Q)$ is compact for $k < 2$, there exists in $S(1)$ the finite ε°-mesh $\{u_n\}_{n=1}^N \subset H_0^k(Q)$ $(N = N(\varepsilon^\circ) < \infty)$. By the conditions imposed on the operators M_ε (1.2), given any $u \in S(1)$, there is an element u_n of the ε°-mesh such that

$$|\langle M_\varepsilon u, u\rangle_2 - \langle M_\varepsilon u_n, u_n\rangle_2| \leqslant C\varepsilon^\circ \tag{3.4}$$

where the constant $C < \infty$ on the right-hand side can be chosen independently of $u \in S(1)$. Moreover, since the sequence of operators M_ε is strongly convergent to the operator M in $H^{-k}(Q)$ and the ε°-mesh is finite for any ε, starting with some $\varepsilon_0(N) > 0$, we have

$$|\langle M_\varepsilon u_n, u_n\rangle_2 - \langle Mu_n, u_n\rangle_2| \leqslant \varepsilon^\circ \tag{3.5}$$

$$\forall u_n \in \{u_n\}_{n=1}^N, \quad N = N(\varepsilon^\circ) < \infty$$

We consider $\delta > 0$ and choose ε° from the condition $C\varepsilon^\circ < \delta/3$ (see (3.4)), then choose $\varepsilon_0(N) > 0$ from the condition $\varepsilon^\circ < \delta/3$ (see (3.5)). Then, by (3.4) and (3.5), and the inequality of type (3.4) for operator M, we see that, for all $u \in S(1)$ and $\varepsilon \leqslant \varepsilon_0(N)$, the left-hand side of (3.4) is not greater than δ. From this, and the estimate for $\langle L_\varepsilon u, u\rangle_2$, we obtain (3.3).

Lemma 7. Starting with some $\varepsilon_0(\delta) > 0$, we have

$$\langle L_\varepsilon u, u\rangle_2 - \langle M_\varepsilon u, u\rangle_2 \geqslant \mu \|u\|_2^2, \ \forall u \in H_0^2(Q) \tag{3.6}$$

where μ is the right-hand side of (3.3).

The lemma is obvious, since the operators L_ε, M_ε are linear.

Proposition 4. If the tensor $\sigma_{ij}(\mathbf{x})$ of the average stressed state is non-positive definite, then, starting from some $\varepsilon' > 0$, the operators $L_\varepsilon - M_\varepsilon$ are positive definite for all $0 < \varepsilon \leqslant \varepsilon'$.

Proof. In the present case

$$\langle -Mu, u\rangle_2 = -\int\limits_Q \sigma_{ij}(\mathbf{x})\, u_{,i} u_{,j}\, d\mathbf{x} \geqslant 0, \quad \forall u \in H_0^2(Q)$$

in view of which it is sufficient to use Lemmas 6 and 7, putting $\delta < c/2$ in (3.3).

We can now use Theorem 6.3 of /17, p.89/ to study the asymptotic behaviour as $\varepsilon \to 0$ of the solution of problem (3.1), (3.2). For the theorem to be applicable, the stationary part of problem (3.1), (3.2), i.e., the operators $L_\varepsilon - M_\varepsilon$, must be positive definite, which is the case here by Proposition 4.

Proposition 5. Let the sequences of operators L_ε, M_ε be convergent to the operators L, M in the sense indicated in Proposition 1, and let the tensor $\sigma_{ij}(\mathbf{x})$ be non-positive definite. Then, the sequence of solutions of problem (3.1), (3.2) $w^\varepsilon \to w$ *-weakly in $L_\infty([0, \infty], H_0^2(Q))$ as $\varepsilon \to 0$, where w is the solution of the equation

$$-Lw + Mw = \langle\rho\rangle(\mathbf{x})\partial^2 w/\partial t^2 + f(\mathbf{x})$$

with initial conditions (3.2).

The proof is a repetition of the proof of Theorem 6.3 of /17/ (with suitable replacement of the functional spaces, since part of the results in /17/ is obtained in the context of a second-order equation).

Note 4. It is interesting to study problem (3.1), (3.2) in the case when the operators are non-positive (e.g., oscillations of compressed plate). We showed above that the problem of the natural oscillations admits of an averaged description in this case, admittedly under the condition (condition A) that the stresses in the plane of the plate do not lead to loss of stability; this further poses the question of an average description of the oscillations when the plate stability is lost or is near to being lost.

Note 5. Our results of Paras.1 and 2 can be extended to the case of densely perforated plates. In the proofs, we then have to use, in addition to our above methods, the methods and results of /18, 19/. By using the results of /20/ we can obtain here explicit expressions for the averaged characteristics of reticular plates.

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